

Optical theorem for Aharonov-Bohm scattering

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Abstract

Quantum-mechanical scattering off a magnetic vortex is considered, and the optical theorem is derived. The vortex core is assumed to be impermeable to scattered particles, and its transverse size is taken into account. We show that the scattering Aharonov-Bohm effect is independent of the choice of boundary conditions from the variety of the Robin ones. The behaviour of the scattering amplitude in the forward direction is analyzed, and the persistence of the Fraunhofer diffraction in the short-wavelength limit is shown to be crucial for maintaining the optical theorem in the quasiclassical limit.

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1 Introduction

Probability conservation and the unitarity of the scattering matrix are the basic elements of quantum theory, which have significant physical consequences. One of them is the optical theorem relating the imaginary part of the scattering amplitude in the forward direction to the total cross section of the interaction

processes. Quantum-mechanical scattering of a charged particle off an impermeable magnetic vortex is studied for a more than half a century, starting from the seminal paper of Aharonov and Bohm [1]. The theory of this process has been successfully confirmed in experiments, promising important practical applications, see review [2]. However, some theoretical issues still remain unclear, and among them the question on how to formulate the optical theorem for the case of the Aharonov-Bohm scattering. Several authors [3, 4] addressed this problem but without any decisive conclusion (see also [5]); they traced the encountered difficulties either to a subtlety in the choice of an incident wave [3], or to a divergent behaviour of the scattering amplitude in the forward direction, which needs yet unspecified regularization [4]. In our opinion, the following two circumstances should be taken into consideration: a) the long-range nature of interaction of a scattering particle with the magnetic vortex, and b) the nonvanishing transverse size of the magnetic vortex. Due to the first circumstance, although the unitarity of the S -matrix is undoubted, the standard scattering theory which is applicable to the case of short-range interactions has to be modified, resulting in a rather unexpected form of the optical theorem. The second circumstance signifies that the limit of the vanishing transverse size of the vortex is an undue idealization which has to be avoided as physically irrelevant. Then, provided that points a) and b) are taken into account, the above problem can be treated properly and resolved, as it is shown in the present paper.

The magnetic field configuration in the form of an infinitely long vortex possesses cylindrical symmetry. The S -matrix in the cylindrically symmetric case takes form

$$S(k, \varphi, k_z; k', \varphi', k'_z) = \left[I(k, \varphi; k', \varphi') + \delta(k - k') \frac{i}{\sqrt{2\pi k}} f(k, \varphi - \varphi') \right] \delta(k_z - k'_z), \quad (1)$$

where the symmetry axis coincides with the z -axis, $I(k, \varphi; k', \varphi')$ is the unity matrix in polar coordinates in two-dimensional space, and $f(k, \varphi - \varphi')$ is the scattering amplitude. The condition of the unitarity of the S -matrix, $S^\dagger S = SS^\dagger = I$, results, in a conventional way, in the optical theorem

$$2\sqrt{\frac{2\pi}{k}} \text{Im} f(k, 0) = \sigma, \quad (2)$$

where σ is the total cross section per unit length along the symmetry axis, and, thus, σ has the dimension of length. We shall prove that, namely in the case of the Aharonov-Bohm scattering, the optical theorem takes a form which is different from the conventional one given by (2). Our consideration is based on earlier works [6, 7, 8, 9, 10, 11] where important results concerning the scattering wave function, S -matrix and scattering amplitude have been obtained.

In the next section we review an auxiliary problem of scattering off an impermeable tube; the role of the Fraunhofer diffraction and the appropriate forward peak is exposed. In Section 3 we consider the same problem for the case when the tube is filled with the magnetic flux lines, i.e. for the case of an impermeable magnetic vortex; the main results are derived here. Summary and discussion of the results are given in Section 4. We relegate the details of calculation of the scattering amplitude and the total cross section in the quasiclassical limit to Appendices A and B.

2 Scattering off an impermeable tube

A plane wave propagating in the direction which is orthogonal to the z -axis can be presented as

$$\begin{aligned}\psi_{\mathbf{k}}^{(0)}(\mathbf{r}) &= e^{ikr \cos \varphi} = \sum_{n \in \mathbb{Z}} e^{in\varphi} e^{\frac{i}{2}|n|\pi} J_{|n|}(kr) = \\ &= \frac{1}{2} \sum_{n \in \mathbb{Z}} e^{in\varphi} e^{\frac{i}{2}|n|\pi} \left[H_{|n|}^{(1)}(kr) + H_{|n|}^{(2)}(kr) \right],\end{aligned}\quad (3)$$

where \mathbf{r} and \mathbf{k} are the two-dimensional vectors, φ is the angle between them, $J_\alpha(u)$, $H_\alpha^{(1)}(u)$ and $H_\alpha^{(2)}(u)$ are the Bessel, first- and second-kind Hankel functions of order α , \mathbb{Z} is the set of integer numbers. At $r \rightarrow \infty$, using the appropriate asymptotics of the Hankel functions, one gets

$$\begin{aligned}\psi_{\mathbf{k}}^{(0)}(\mathbf{r}) &\underset{r \rightarrow \infty}{\sim} \frac{1}{\sqrt{2\pi kr}} \sum_{n \in \mathbb{Z}} e^{in\varphi} e^{\frac{i}{2}|n|\pi} \left[e^{i(kr - \frac{1}{2}|n|\pi - \frac{1}{4}\pi)} + \right. \\ &\left. + e^{-i(kr - \frac{1}{2}|n|\pi - \frac{1}{4}\pi)} \right] = \sqrt{\frac{2\pi}{kr}} e^{i(kr - \pi/4)} \Delta(\varphi) + \sqrt{\frac{2\pi}{kr}} e^{-i(kr - \pi/4)} \Delta(\varphi - \pi),\end{aligned}\quad (4)$$

where

$$\Delta(\varphi) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in\varphi} \quad (5)$$

is the delta-function for the azimuthal angle, $\Delta(\varphi + 2\pi) = \Delta(\varphi)$. Thus, we see that the plane wave passing through the origin ($r = 0$) can be naturally interpreted at large distances from the origin as a superposition of two cylindrical waves: the diverging one, e^{ikr} , in the forward, $\varphi = 0$, direction and the converging one, e^{-ikr} , from the backward, $\varphi = \pi$, direction.

Now, let us place an obstacle in the form of an opaque tube along the z -axis. If the wave function obeys the Dirichlet boundary condition at the edge of the tube,

$$\psi_{\mathbf{k}}(\mathbf{r})|_{r=r_c} = 0, \quad (6)$$

then, instead of (3), we get

$$\psi_{\mathbf{k}}(\mathbf{r}) = \sum_{n \in \mathbb{Z}} e^{in\varphi} e^{\frac{i}{2}|n|\pi} \left[J_{|n|}(kr) - \frac{J_{|n|}(kr_c)}{H_{|n|}^{(1)}(kr_c)} H_{|n|}^{(1)}(kr) \right]. \quad (7)$$

At large distances from the origin, $r \gg k^{-1}$, we get

$$\psi_{\mathbf{k}}(\mathbf{r}) \underset{kr \gg 1}{=} - \frac{1}{\sqrt{2\pi kr}} e^{i(kr - \pi/4)} \sum_{n \in \mathbb{Z}} e^{in\varphi} \frac{H_{|n|}^{(2)}(kr_c)}{H_{|n|}^{(1)}(kr_c)} + \sqrt{\frac{2\pi}{kr}} e^{-i(kr - \pi/4)} \Delta(\varphi - \pi). \quad (8)$$

The sum over n in (8) yields the forward angular delta-function, $\Delta(\varphi)$, in the case of long wavelengths, $k \rightarrow 0$. In the opposite, short-wavelength, limit, $k \rightarrow \infty$, when $1 \ll kr_c < kr$, one can get by substituting the large-argument asymptotics of the Hankel functions in (8):

$$\psi_{\mathbf{k}}(\mathbf{r})_{kr > kr_c \gg 1} = \sqrt{\frac{2\pi}{kr}} e^{i\pi/4} [e^{ik(r-2r_c)} - e^{-ikr}] \Delta(\varphi - \pi). \quad (9)$$

This result which is actually given in the monograph of Morse and Feshbach [12] is quite understandable from the classical point of view: the obstacle forms a shadow in the forward direction, which is not accessible to waves, and, thus, both diverging and converging cylindrical waves are in the backward direction. However, this conclusion is wrong.

To find a loophole in the arguments leading to (9), one has to note a property of the asymptotical behaviour of the Bessel function at large values of its argument: it vanishes effectively when its order exceeds its large argument. Really, using integral representation (see, e.g. [13])

$$J_{|n|}(u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \exp[i(|n|\theta - u \sin \theta)],$$

one notes that the integrand at large $|n|$ is vigorously oscillating and its mean value is small almost everywhere with the exception of points where the phase is stationary. This means that the prevailing contribution to the integral in the case of $|n| \gg 1$ is given by the vicinity of the point where $\cos \theta = |n|/u$, and, consequently, the integral is vanishingly small in the case of $|n|/u \gg 1$. The more is u , the more abrupt is the decrease of the integral when $|n|$ exceeds u (see, e.g., [14]).

Therefore, in (7) at $kr > kr_c \gg 1$, the sum containing $J_{|n|}(kr)$ is cut at $|n| = kr$, while the sum containing $J_{|n|}(kr_c)$ is cut at $|n| = kr_c$. Instead of (9),

we get the correct expression:

$$\begin{aligned} & \psi_{\mathbf{k}}(\mathbf{r})_{kr > \overline{kr}_c \gg 1} \sqrt{\frac{2\pi}{kr}} e^{i(kr - \pi/4)} [\Delta_{kr}(\varphi) - \Delta_{kr_c}(\varphi)] + \\ & + \sqrt{\frac{2\pi}{kr}} e^{i\pi/4} [e^{-ikr} \Delta_{kr}(\varphi - \pi) - e^{ik(r-2r_c)} \Delta_{kr_c}(\varphi - \pi)], \end{aligned} \quad (10)$$

where

$$\Delta_x(\varphi) = \frac{1}{2\pi} \sum_{|n| \leq x} e^{in\varphi} \quad (11)$$

is the regularized (smoothed) angular delta-function,

$$\lim_{x \rightarrow \infty} \Delta_x(\varphi) = \Delta(\varphi), \quad \Delta_x(0) = \frac{x}{\pi}. \quad (12)$$

We see that a cancellation of the diverging wave in the forward direction is not complete, as well as is that between the diverging and the converging waves in the backward direction; the complete cancellation is achieved at $r = r_c$ only, that is consistent with condition (6). In general, the diverging wave in the forward direction is suppressed by factor $1 - r_c r^{-1}$, as compared to the case when the obstacle is absent. Thus, contrary to the classical anticipations, the wave penetrates to the region behind the obstacle even in the case when the wavelength is much less than the transverse size of the obstacle, $kr_c \gg 1$.

Turning now from the qualitative analysis to the quantitative one, we note, first, from (7) that the asymptotics of the wave function at large distances from the obstacle is

$$\psi_{\mathbf{k}}(\mathbf{r}) = \psi_{\mathbf{k}}^{(0)}(\mathbf{r}) + f(k, \varphi) \frac{e^{i(kr + \pi/4)}}{\sqrt{r}} + O(r^{-3/2}), \quad (13)$$

where

$$f(k, \varphi) = i \sqrt{\frac{2}{k\pi}} \sum_{n \in \mathbb{Z}} e^{in\varphi} \frac{J_{|n|}(kr_c)}{H_{|n|}^{(1)}(kr_c)} \quad (14)$$

is the scattering amplitude which enters S -matrix (1), while the unity matrix there is evidently

$$I(k, \varphi; k', \varphi') = \frac{1}{k} \delta(k - k') \Delta(\varphi - \varphi'). \quad (15)$$

It should be noted that wave function (7), apart from condition (6), satisfies also condition

$$\lim_{r \rightarrow \infty} e^{ikr} \psi_{\mathbf{k}}(\mathbf{r})|_{\varphi = \pm\pi} = 1, \quad (16)$$

signifying that the incident wave comes from the far left; the forward direction is $\varphi = 0$, and the backward direction is $\varphi = \pm\pi$.

The S -matrix is unitary:

$$\begin{aligned} \int_{-\infty}^{\infty} dk_z \int_0^{\infty} dk k \int_{-\pi}^{\pi} d\varphi S^*(k, \varphi, k_z; k', \varphi', k'_z) S(k, \varphi, k_z; k'', \varphi'', k''_z) = \\ = \frac{1}{k'} \delta(k' - k'') \Delta(\varphi' - \varphi'') \delta(k'_z - k''_z), \end{aligned} \quad (17)$$

and the latter relation can be recast into the form

$$\frac{1}{i} \sqrt{\frac{k}{2\pi}} [f(k, \varphi' - \varphi'') - f^*(k, \varphi'' - \varphi')] = \frac{k}{2\pi} \int_{-\pi}^{\pi} d\varphi f^*(k, \varphi - \varphi') f(k, \varphi - \varphi''), \quad (18)$$

in particular, at $\varphi' = \varphi'' = 0$ one gets optical theorem (2) with $\sigma = \int_{-\pi}^{\pi} d\varphi |f(k, \varphi)|^2$ being the total cross section for elastic scattering. Although (18) is valid for all wavelengths, it relates vanishingly small quantities in the long-wavelength limit, whereas in the short-wavelength limit it relates extremely large quantities.

Really, the scattering amplitude in the long-wavelength limit is estimated as

$$f(k, \varphi) = -\sqrt{\frac{\pi}{2k}} |\ln(kr_c)|^{-1} \left[1 + \left(\gamma - i\frac{\pi}{2} \right) |\ln(kr_c)|^{-1} \right] + k^{-1/2} O[|\ln(kr_c)|^{-3}], \quad kr_c \ll 1 \quad (19)$$

(γ is the Euler constant), and (18) is the relation between quantities of order $O[|\ln(kr_c)|^{-2}]$. The scattering amplitude in the short-wavelength limit is

$$f(k, \varphi) = i\sqrt{\frac{2\pi}{k}} \Delta_{kr_c}(\varphi) - \sqrt{\frac{r_c}{2}} |\sin(\varphi/2)| \exp\{-2ikr_c |\sin(\varphi/2)|\} e^{-i\pi/4}, \quad kr_c \gg 1. \quad (20)$$

The first term in (20) represents the forward peak of the Fraunhofer diffraction on the obstacle, whereas the second term describes the reflection from the obstacle according to the laws of geometric (ray) optics. Hence, (18) in this case is the relation between quantities of order $O(kr_c)$ at $\varphi' = \varphi''$ or of order $O(\sqrt{kr_c})$ at $\varphi' \neq \varphi''$, and optical theorem (2) is the relation between finite quantities of order r_c . It should be noted that the Fraunhofer diffraction is crucial for ensuring the optical theorem in the short-wavelength limit, since the second term in (20) vanishes at $\varphi = 0$.

3 Scattering off an impermeable magnetic vortex

Let us consider scattering of a charged particle off an obstacle in the form of an impermeable tube which is filled with magnetic field of total flux Φ . The particle wave function obeys conditions (6) and (16); an additional circumstance is that the Schrödinger hamiltonian out of the tube is no longer free but takes form

$$H = -\frac{\hbar^2}{2m} \left[\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \left(\partial_\varphi - i \frac{\Phi}{\Phi_0} \right)^2 + \partial_z^2 \right], \quad (21)$$

where $\Phi_0 = 2\pi\hbar ce^{-1}$ is the London flux quantum. Therefore, the scattering wave solution in this case is (cf. (7))

$$\psi_{\mathbf{k}}(\mathbf{r}) = \sum_{n \in \mathbb{Z}} e^{in\varphi} e^{i(|n| - \frac{1}{2}|n-\mu|)\pi} \left[J_{|n-\mu|}(kr) - \frac{J_{|n-\mu|}(kr_c)}{H_{|n-\mu|}^{(1)}(kr_c)} H_{|n-\mu|}^{(1)}(kr) \right], \quad (22)$$

where $\mu = \Phi\Phi_0^{-1}$.

In the long-wavelength limit, we get the asymptotics

$$\begin{aligned} \psi_{\mathbf{k}}(\mathbf{r}) \underset{\substack{kr \gg 1 \\ kr_c \ll 1}}{\approx} & \sqrt{\frac{2\pi}{kr}} e^{i(kr - \pi/4)} [\cos(\mu\pi) \Delta(\varphi) - \sin(\mu\pi) \Gamma^{(\nu)}(\varphi)] + \\ & + \sqrt{\frac{2\pi}{kr}} e^{-i(kr - \pi/4)} \Delta(\varphi - \pi), \end{aligned} \quad (23)$$

where $\nu = \llbracket \mu \rrbracket$ and $\llbracket u \rrbracket$ denotes the integer part of u (i.e. the integer which is less than or equal to u),

$$\Gamma^{(\nu)}(\varphi) = \frac{1}{2\pi i} \sum_{n \in \mathbb{Z}} \text{sgn}(n - \mu) e^{in\varphi} \quad (24)$$

and $\text{sgn}(u) = \begin{cases} 1, & u > 0 \\ -1, & u < 0 \end{cases}$. In the short-wavelength limit, we get the asymptotics

$$\begin{aligned} \psi_{\mathbf{k}}(\mathbf{r}) \underset{kr > \overline{kr_c} \gg 1}{\approx} & \sqrt{\frac{2\pi}{kr}} e^{i(kr - \pi/4)} \left\{ \cos(\mu\pi) \left[\Delta_{kr}^{(\nu)}(\varphi) - \Delta_{kr_c}^{(\nu)}(\varphi) \right] - \right. \\ & \left. - \sin(\mu\pi) \left[\Gamma_{kr}^{(\nu)}(\varphi) - \Gamma_{kr_c}^{(\nu)}(\varphi) \right] \right\} + \sqrt{\frac{2\pi}{kr}} e^{i\pi/4} \left[e^{-ikr} \Delta_{kr}^{(\nu)}(\varphi - \pi) - e^{ik(r - 2r_c)} \Delta_{kr_c}^{(\nu)}(\varphi - \pi) \right], \end{aligned} \quad (25)$$

where

$$\Delta_x^{(\nu)}(\varphi) = \frac{1}{2\pi} \sum_{|n-\mu| \leq x} e^{in\varphi}, \quad \Gamma_x^{(\nu)}(\varphi) = \frac{1}{2\pi i} \sum_{|n-\mu| \leq x} \text{sgn}(n - \mu) e^{in\varphi}; \quad (26)$$

note that $\Delta_x^{(\nu)}(\varphi)$ can be regarded as a regularization for $\Delta(\varphi)$ (5), since $\Delta_x^{(\nu)}(\varphi)$, as well as $\Delta_x(\varphi)$ (11), satisfies conditions (12). It should be also noted that, since it is a merely qualitative analysis, the long-wavelength asymptotics of the wave function can be estimated as well as (23) with $\Delta_{kr}^{(\nu)}$ and $\Gamma_{kr}^{(\nu)}$ substituted for Δ and $\Gamma^{(\nu)}$.

Thus, we conclude that, both in the long- and short-wavelength limits the particle wave penetrating in the forward direction behind the obstacle depends periodically as cosine on the enclosed magnetic flux with the period equal to the London flux quantum. This periodic dependence, as well as the sine periodic dependence of the reflected wave in other directions, is due to the fact that the interaction with the scatterer is neither of the potential type, nor of the sufficient decrease at large distances from the scatterer, see hamiltonian (21).

Turning now from the qualitative analysis to the quantitative one, we note, first, that, owing to the long-range nature of interaction, the unity matrix in (1) is distorted:

$$I(k, \varphi; k', \varphi') = \cos(\mu\pi) \frac{1}{k} \delta(k - k') \Delta(\varphi - \varphi'), \quad (27)$$

while the scattering amplitude is

$$f(k, \varphi) = f_0(k, \varphi) + f_c(k, \varphi), \quad (28)$$

where

$$f_0(k, \varphi) = i\sqrt{\frac{2\pi}{k}} \sin(\mu\pi) \Gamma^{(\nu)}(\varphi) \quad (29)$$

and

$$f_c(k, \varphi) = i\sqrt{\frac{2}{k\pi}} \sum_{n \in \mathbb{Z}} e^{in\varphi} e^{i(|n| - |n-\mu|)\pi} \frac{J_{|n-\mu|}(kr_c)}{H_{|n-\mu|}^{(1)}(kr_c)}. \quad (30)$$

The asymptotics of the wave function at large distances from the vortex is

$$\psi_{\mathbf{k}}(\mathbf{r}) = \psi_{\mathbf{k}}^{(0)}(\mathbf{r}) e^{i\mu[\varphi - \text{sgn}(\varphi)\pi]} + f(k, \varphi) \frac{e^{i(kr + \pi/4)}}{\sqrt{r}} + O(r^{-3/2}), \quad (31)$$

where it is implied that $-\pi < \varphi < \pi$.

From now on, we extend our consideration to involve a more general boundary condition for the wave function at the edge of the impermeable magnetic vortex. Namely, we impose the Robin boundary condition,

$$\left\{ \left[\cos(\rho\pi) + r_c \sin(\rho\pi) \frac{\partial}{\partial r} \right] \psi_{\mathbf{k}}(\mathbf{r}) \right\} \Big|_{r=r_c} = 0, \quad (32)$$

then (22) is changed to

$$\psi_{\mathbf{k}}(\mathbf{r}) = \sum_{n \in \mathbb{Z}} e^{in\varphi} e^{i(|n| - \frac{1}{2}|n-\mu|)\pi} \left[J_{|n-\mu|}(kr) - \Upsilon_{|n-\mu|}^{(\rho)}(kr_c) H_{|n-\mu|}^{(1)}(kr) \right], \quad (33)$$

and (30) is changed to

$$f_c(k, \varphi) = i\sqrt{\frac{2}{k\pi}} \sum_{n \in \mathbb{Z}} e^{in\varphi} e^{i(|n| - |n-\mu|\pi)} \Upsilon_{|n-\mu|}^{(\rho)}(kr_c), \quad (34)$$

where

$$\Upsilon_{\alpha}^{(\rho)}(u) = \frac{\cos(\rho\pi)J_{\alpha}(u) + \sin(\rho\pi)u \frac{d}{du}J_{\alpha}(u)}{\cos(\rho\pi)H_{\alpha}^{(1)}(u) + \sin(\rho\pi)u \frac{d}{du}H_{\alpha}^{(1)}(u)}. \quad (35)$$

Hence, $\rho = 0$ corresponds to the Dirichlet condition (perfect conductivity of the boundary), see (6), and $\rho = 1/2$ corresponds to the Neumann condition (absolute rigidity of the boundary),

$$\left[\frac{\partial}{\partial r} \psi_{\mathbf{k}}(\mathbf{r}) \right] \Big|_{r=r_c} = 0. \quad (36)$$

3.1 Optical theorem

The unitarity condition for the S -matrix, (17), is certainly valid for all wavelengths. However, due to the long-range nature of interaction, the consequent condition in terms of the scattering amplitude takes forms which differ from (18).

The long-wavelength limit, $k \rightarrow 0$, is the same as the $r_c \rightarrow 0$ limit corresponding to the idealized case of a singular vortex of zero thickness. Amplitude f_c (34) can be neglected in this case, and the S -matrix unitarity condition involves amplitude f_0 (29) only. In view of (27) and relation

$$\Gamma^{(\nu)}(\varphi) + [\Gamma^{(\nu)}(-\varphi)]^* = 0, \quad (37)$$

we get, instead of (18), the following relation:

$$\sin^2(\mu\pi)\Delta(\varphi' - \varphi'') = \frac{k}{2\pi} \int_{-\pi}^{\pi} d\varphi f_0^*(k, \varphi - \varphi') f_0(k, \varphi - \varphi''). \quad (38)$$

Thus, we see that the optical theorem which should be derived from (38) by putting $\varphi' = \varphi'' = 0$ is hardly informative, being a relation between infinite quantities, $\Delta(0)$.

The failure with the optical theorem for the Aharonov-Bohm scattering in the long-wavelength limit is due to an unphysical idealization inherent in the treatment of this case. As long as the transverse size of the vortex is taken into account, the optical theorem emerges as a relation between finite quantities.

Really, retaining f_c in the unitarity relation (17), we get

$$\begin{aligned} \frac{1}{i} \sqrt{\frac{k}{2\pi}} \left\{ \cos(\mu\pi) [f_c(k, \varphi' - \varphi'') - f_c^*(k, \varphi'' - \varphi')] + \right. \\ \left. + \sin(\mu\pi) \int_{-\pi}^{\pi} d\varphi [\Gamma^{(\nu)}(\varphi' - \varphi) f_c(k, \varphi - \varphi'') + \right. \\ \left. + f_c^*(k, \varphi - \varphi') \Gamma^{(\nu)}(\varphi - \varphi'')] \right\} = \frac{k}{2\pi} \int_{-\pi}^{\pi} d\varphi f_c^*(k, \varphi - \varphi') f_c(k, \varphi - \varphi''), \quad (39) \end{aligned}$$

which at $\varphi' = \varphi'' = 0$ yields optical theorem

$$\begin{aligned} \sqrt{\frac{2\pi}{k}} \left\{ 2 \cos(\mu\pi) \text{Im} f_c(k, 0) - i \sin(\mu\pi) \int_{-\pi}^{\pi} d\varphi [\Gamma^{(\nu)}(-\varphi) f_c(k, \varphi) + \right. \\ \left. + \Gamma^{(\nu)}(\varphi) f_c^*(k, \varphi)] \right\} = \int_{-\pi}^{\pi} d\varphi |f_c(k, \varphi)|^2. \quad (40) \end{aligned}$$

As the wavelength decreases, f_0 is decreasing as $O(k^{-1/2})$, see (29), becoming negligible as compared to f_c . Thus, the right-hand side of (40) tends to the total cross section in the short-wavelength limit. Further, estimating appropriately the integral in the left-hand side of (39), we find that this relation in the short-wavelength limit turns out to be

$$\begin{aligned} \frac{1}{i} \sqrt{\frac{k}{2\pi}} \cos(\mu\pi) [f_c(k, \varphi' - \varphi'') - f_c^*(k, \varphi'' - \varphi')] + 2 \sin^2(\mu\pi) \Delta_{kr_c}^{(\nu)}(\varphi' - \varphi'') + \\ + O(\sqrt{kr_c}) = \frac{k}{2\pi} \int_{-\pi}^{\pi} d\varphi f_c^*(k, \varphi - \varphi') f_c(k, \varphi - \varphi''), \quad (41) \end{aligned}$$

and the optical theorem in this limit takes form

$$2 \sqrt{\frac{2\pi}{k}} \cos(\mu\pi) \text{Im} f_c(k, 0) + \frac{4\pi}{k} \sin^2(\mu\pi) \Delta_{kr_c}^{(\nu)}(0) + O(k^{-1}) = \sigma, \quad (42)$$

where

$$\sigma = \int_{-\pi}^{\pi} d\varphi |f_c(k, \varphi)|^2, \quad kr_c \gg 1. \quad (43)$$

3.2 Alternative decomposition of S -matrix

The S -matrix can be presented as

$$S(k, \varphi, k_z; k', \varphi', k'_z) = \left[I(k, \varphi; k', \varphi') + \delta(k - k') \frac{i}{\sqrt{2\pi k}} \tilde{f}(k, \varphi - \varphi') \right] \delta(k_z - k'_z), \quad (44)$$

where $I(k, \varphi; k', \varphi')$ is the usual unity matrix given by (15), and

$$\tilde{f}(k, \varphi) = i\sqrt{\frac{2\pi}{k}} [1 - \cos(\mu\pi)] \Delta(\varphi) + f(k, \varphi); \quad (45)$$

then the asymptotics of the wave function can be rewritten as, cf. (31),

$$\psi_{\mathbf{k}}(\mathbf{r}) = \psi_{\mathbf{k}}^{(0)}(\mathbf{r}) + \tilde{f}(k, \varphi) \frac{e^{i(kr + \pi/4)}}{\sqrt{r}} + O(r^{-3/2}). \quad (46)$$

Such a decomposition was proposed in [7], and the arguments in its favour were given, for instance, in [3]. Although the unitarity condition in terms of amplitude \tilde{f} takes the conventional form given by (18), the question is whether it relates physically meaningful (and thus finite) quantities. The answer is clearly negative.

Indeed, one can easily get

$$\begin{aligned} \frac{k}{2\pi} \int_{-\pi}^{\pi} d\varphi \tilde{f}^*(k, \varphi - \varphi') \tilde{f}(k, \varphi - \varphi'') &= 2[1 - \cos(\mu\pi)] \Delta(\varphi' - \varphi'') - \\ &- i\sqrt{\frac{k}{2\pi}} [f_c(k, \varphi' - \varphi'') - f_c^*(k, \varphi'' - \varphi')], \end{aligned} \quad (47)$$

and, consequently, the total cross section in this framework,

$$\tilde{\sigma} = \int_{-\pi}^{\pi} d\varphi |\tilde{f}(k, \varphi)|^2, \quad (48)$$

contains an unremovable divergency (actual infinity), $\frac{4\pi}{k} [1 - \cos(\mu\pi)] \Delta(0)$, for all wavelengths.

On the contrary, in the present paper we argue for a more physically plausible framework. The total cross section in the long-wavelength limit contains unremovable divergency, $\frac{2\pi}{k} \sin^2(\mu\pi) \Delta(0)$ (see (38)), and this divergency is due to an unphysical idealization involving singular vortex. However, the total cross section in the short-wavelength limit is given by (43) which is finite and, as it will be shown in the following, independent of both flux (μ) and boundary (ρ) parameters.

3.3 Scattering amplitude

The scattering amplitude in the short-wavelength limit has been obtained in [15] for the case of the Dirichlet boundary condition. In Appendix A it is evaluated for the case of the Robin boundary condition:

$$f_c(k, \varphi) = i\sqrt{\frac{2\pi}{k}} \left[\cos(\mu\pi) \Delta_{kr_c}^{(\nu)}(\varphi) - \sin(\mu\pi) \Gamma_{kr_c}^{(\nu)}(\varphi) \right] - \sqrt{\frac{r_c}{2} |\sin(\varphi/2)|} \times \\ \times \exp \left\{ -2ikr_c |\sin(\varphi/2)| + i\mu[\varphi - \text{sgn}(\varphi)\pi] \right\} \exp \left\{ -i \left[2\chi^{(\rho)}(kr_c, \varphi) + \pi/4 \right] \right\} + \\ + \sqrt{r_c} O \left[(kr_c)^{-1/6} \right], \quad kr_c \gg 1, \quad (49)$$

where

$$\chi^{(\rho)}(kr_c, \varphi) = \arctan \left[\frac{2kr_c |\sin^3(\varphi/2)|}{2\cot(\rho\pi) \sin^2(\varphi/2) - 1} \right]. \quad (50)$$

The Fraunhofer diffraction on the vortex in the forward direction is described by the first term, while the classical reflection from the vortex in other directions is described by the second term which, apart from the phase factor, is the same as the second term in (20). The explicit form of $\Delta_{kr_c}^{(\nu)}(\varphi)$ and $\Gamma_{kr_c}^{(\nu)}(\varphi)$ is as follows:

$$\Delta_{kr_c}^{(\nu)}(\varphi) = \frac{e^{i(\nu+\frac{1}{2})\varphi}}{2\pi} \frac{\sin(s_c\varphi)}{\sin(\varphi/2)} \quad (51)$$

and

$$\Gamma_{kr_c}^{(\nu)}(\varphi) = \frac{e^{i(\nu+\frac{1}{2})\varphi}}{2\pi} \frac{1 - \cos(s_c\varphi)}{\sin(\varphi/2)} \quad (52)$$

in the case

$$\llbracket kr_c + \mu \rrbracket - \nu = \llbracket kr_c - \mu \rrbracket + \nu + 1 = s_c, \quad (53)$$

or

$$\Delta_{kr_c}^{(\nu)}(\varphi) = \frac{e^{i(\nu+\frac{1}{2}\mp\frac{1}{2})\varphi}}{2\pi} \frac{\sin \left[\left(s_c + \frac{1}{2} \right) \varphi \right]}{\sin(\varphi/2)} \quad (54)$$

and

$$\Gamma_{kr_c}^{(\nu)}(\varphi) = \frac{e^{i(\nu+\frac{1}{2}\mp\frac{1}{2})\varphi}}{2\pi} \left\{ \frac{1 - \cos \left[\left(s_c + \frac{1}{2} \right) \varphi \right]}{\sin(\varphi/2)} - \text{tg}(\varphi/4) \pm i \right\} \quad (55)$$

in the case

$$\llbracket kr_c + \mu \rrbracket - \nu - \frac{1}{2} \pm \frac{1}{2} = \llbracket kr_c - \mu \rrbracket + \nu + \frac{1}{2} \mp \frac{1}{2} = s_c. \quad (56)$$

Thus, in the short-wavelength limit, we get in the strictly forward direction:

$$f_c(k, 0) = i\sqrt{\frac{2k}{\pi}} r_c \cos(\mu\pi) + O(k^{-1/2}). \quad (57)$$

It should be noted that the left-hand side of (42) in the nonvanishing order involves the contribution of the diffraction peak only, whereas the right-hand side of (42) includes the contribution of the classical reflection as well. In Appendix B we show that the total cross section in the short-wavelength limit is independent of the vortex flux, as well as of the choice of the boundary condition from the variety of the Robin ones:

$$\sigma = 4r_c + O(k^{-1}), \quad (58)$$

this is twice the classical total cross section, the latter been equal to $2r_c$. Already the differential cross section in the short-wavelength limit is independent of the choice of the boundary condition:

$$\begin{aligned} \frac{d\sigma}{d\varphi} &= |f_c(k, \varphi)|^2 = 2r_c \left\{ \cos(2\mu\pi) \tilde{\Delta}_{kr_c}(\varphi) + \right. \\ &+ [1 - \cos(2\mu\pi) - \sin(2\mu\pi) \sin(kr_c\varphi)] \tilde{\Delta}_{\frac{1}{2}kr_c}(\varphi) \left. \right\} + \frac{r_c}{2} |\sin(\varphi/2)| + r_c O[(kr_c)^{-1/2}], \end{aligned} \quad (59)$$

where

$$\tilde{\Delta}_x(\varphi) = \frac{1}{4\pi x} \frac{\sin^2(x\varphi)}{\sin^2(\varphi/2)} \quad (x \gg 1, -\pi < \varphi < \pi) \quad (60)$$

is a one more regularization of the angular delta-function, satisfying (12). By integrating (59) over all angles, one gets easily (58). A third way to get (58) is to insert (57) into optical theorem (42). Thus, the contribution of the diffraction peak to the total cross section is flux independent and is equal to that of classical reflection. The latter is well known for scattering off a tube with zero magnetic flux, see, e.g., [12], and, for the case of nonzero magnetic flux, it has been first established under the Dirichlet boundary condition [15].

It is instructive to derive the explicit form of $\Gamma^{(\nu)}(\varphi)$ (24) here. Using elementary trigonometric relation

$$\cot(\varphi/2) \{ \sin[(n+1)\varphi] - \sin(n\varphi) \} = \cos[(n+1)\varphi] + \cos(n\varphi),$$

one can get

$$\int_0^\pi d\varphi \cot(\varphi/2) \sin(N\varphi) = \pi, \quad N = 1, 2, 3, \dots,$$

which results in relation

$$\cot(\varphi/2) = 2 \sum_{\substack{n \in \mathbb{Z} \\ n \geq 1}} \sin(n\varphi).$$

The use of the latter along with the definition (5) yields

$$\sum_{\substack{n \in \mathbb{Z} \\ n \geq N}} e^{in\varphi} = \pi \Delta(\varphi) - e^{iN\varphi} (e^{i\varphi} - 1)^{-1}, \quad N = 1, 2, 3, \dots, \quad (61)$$

whence it follows that

$$\Gamma^{(\nu)}(\varphi) = \frac{e^{i(\nu+\frac{1}{2})\varphi}}{2\pi} \frac{1}{\sin(\varphi/2)}, \quad (62)$$

where the divergence at $\varphi = 2\pi l$ ($l \in \mathbb{Z}$) in (62), as well as in the second term in (61), is to be understood in the principal-value sense.

Although amplitude f_0 (29) with $\Gamma^{(\nu)}$ (62) diverges in the forward direction, this divergence has no physical consequences, because there is a crossover to another regime in the strictly forward direction: amplitude f_0 , instead of being proportional to $k^{-1/2}$, becomes, formally, proportional to $r^{1/2}$. This is most easily seen from the following expression which is valid for all scattering angles and has been obtained in [5]:

$$\begin{aligned} f_0(k, \varphi) \frac{e^{i(kr+\pi/4)}}{\sqrt{r}} &= i \sin(\mu\pi) e^{ikr \cos \varphi} e^{i(\nu+\frac{1}{2})\varphi} \times \\ &\times \operatorname{sgn}[\sin(\varphi/2)] \operatorname{erfc} \left[e^{-i\pi/4} \sqrt{2kr} |\sin(\varphi/2)| \right], \end{aligned} \quad (63)$$

where $\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty du e^{-u^2}$ is the complementary error function. In the strictly forward direction one gets a discontinuity,

$$\lim_{\varphi \rightarrow \pm 0} f_0(k, \varphi) \frac{e^{i(kr+\pi/4)}}{\sqrt{r}} = \pm i \sin(\mu\pi) e^{ikr}, \quad (64)$$

which cancels the discontinuity of the incident wave (first term in (31)), see [16, 17, 11, 18, 19]. Consequently, wave function (31) is finite and continuous in the forward direction:

$$\psi_{\mathbf{k}}(\mathbf{r})|_{\varphi=0} = \cos(\mu\pi) e^{ikr} + f_c(k, 0) \frac{e^{i(kr+\pi/4)}}{\sqrt{r}}, \quad (65)$$

that is consistent with its exact expression (33). The appearance of factor $\cos(\mu\pi)$ in the transmitted wave (first term in (65)) can be intuitively understood as a result of self-interference from different sides of the vortex [18, 19]. As it is shown in the present paper, the same factor appears also in the scattered wave (second term in (65)) due to the Fraunhofer diffraction, see (57).

4 Summary and discussion

The S -matrix, its unitarity and the consequent optical theorem for the case of the Aharonov-Bohm scattering are considered in the present paper. Standard scattering theory (see, e.g., [20]) is not applicable here, since the interaction with scatterer is not of short-range type. That is why, although the S -matrix is evidently unitary, its unitarity condition in terms of the scattering amplitude takes a rather unusual form.

In the ultraquantum (long-wavelength, $k \rightarrow 0$) limit when the vortex thickness is neglected, the unitarity condition takes form (38) with no terms which are linear in the scattering amplitude. The scattering amplitude in the ultraquantum limit, see (29) with (62), was first obtained by Aharonov and Bohm [1] and then rederived in the framework of different approaches (perhaps, the one with the use of $\Gamma^{(\nu)}(\varphi)$ (24), presented here, is the most simple and straightforward). As to the behaviour of this amplitude in the forward direction, the only thing that should be borne in mind is that the divergence has to be understood in the principal-value sense. The total cross section in the ultraquantum limit diverges as well, hence the optical theorem in this limit seems to be hardly efficient, being merely a consistency relation between two divergent (infinite) quantities, $\Delta(0)$.

The divergence of the scattering amplitude and the total cross section, as well as the failure with the optical theorem, has no physical meaning, being an artefact of the approximation which neglects the vortex thickness: this is certainly an excessive idealization, whereas any realistic vortex is of finite nonzero thickness. As long as one departs from the ultraquantum limit and the vortex thickness ($2r_c$) is taken into account, the unitarity condition can be formulated as relation (39) involving the r_c -dependent part of the scattering amplitude, f_c (34); the optical theorem relates f_c in the forward direction to the contribution of f_c to the total cross section, see (40).

In the quasiclassical (short-wavelength, $k \rightarrow \infty$) limit, the vortex-thickness effects are prevailing and f_c approximates fairly the whole scattering amplitude. The unitarity condition in this limit takes form (41), and the optical theorem is given by (42). The scattering amplitude in the quasiclassical limit, f_c (49), consists of two parts: the one corresponding to the Fraunhofer diffraction on the vortex in the forward direction and the other one corresponding to the classical reflection from the vortex in all other directions. The latter, apart from the phase factor, is flux independent, whereas the former is essentially flux dependent being periodic in the value of the flux with the period equal to the London flux quantum. We conclude that the persistence of the Fraunhofer diffraction in the short-wavelength limit is crucial for maintaining the optical theorem in the quasiclassical limit, since the classical reflection vanishes in the forward direction.

It should be emphasized that, although only the contribution of the diffraction

peak is involved in the left-hand side of (42) in the nonvanishing order, the right-hand side of (42) includes both the contributions of the diffraction peak and the classical reflection. Separate flux dependent terms in the left-hand side of (42) compensate each other to yield the flux independent right-hand side of (42), which equals the doubled diameter of the vortex, i.e. the doubled classical total cross section.

Thus, quantum-mechanical scattering theory in the quasiclassical limit gives an effect which is alien to classical mechanics: a particle wave penetrates in the forward direction behind an obstacle, and the Fraunhofer diffraction persisting in the short-wavelength limit is an essential ingredient of this effect. The r_c -independent part of the particle wave function (see either (7) or (33)) produces a transmitted wave, while the r_c -dependent part produces a scattered wave which is due to the Fraunhofer diffraction. If the obstacle is an impermeable magnetic vortex, then both the transmitted and the scattered waves are modulated by cosine of the vortex flux in the units of the London flux quantum, see (65) and (57). All this comprises the scattering Aharonov-Bohm effect persisting in the quasiclassical limit [15]. As it is shown in the present paper, this phenomenon is the same for the choice of either Dirichlet boundary condition (6), or Neumann one (36), and, in general, is independent of the choice of boundary conditions from the variety of Robin ones (32). It would be interesting to observe this phenomenon directly in a scattering experiment with short-wavelength, almost classical, particles.

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Appendix A

Let us denote the infinite sum (with factor 2) which enters amplitude f_c (34) by

$$\Sigma(s, \varphi) = 2 \sum_{n \in \mathbb{Z}} e^{in\varphi} e^{i(|n|-|n-\mu|)\pi} \Upsilon_{|n-\mu|}^{(\rho)}(s), \quad (\text{A.1})$$

where $s = kr_c$, and decompose it into three parts

$$\Sigma(s, \varphi) = \Sigma_1(s, \varphi) + \Sigma_2(s, \varphi) + \Sigma_3(s, \varphi), \quad (\text{A.2})$$

where

$$\Sigma_1(s, \varphi) = \sum_{|n-\mu| \leq s} e^{in\varphi} e^{i(|n|-|n-\mu|)\pi}, \quad (\text{A.3})$$

$$\Sigma_2(s, \varphi) = \sum_{|n-\mu| \leq s} e^{in\varphi} e^{i(|n|-|n-\mu|)\pi} \frac{H_{|n-\mu|}^{(2)}(s)}{H_{|n-\mu|}^{(1)}(s)} \frac{\cot(\rho\pi) + s \frac{d}{ds} \ln H_{|n-\mu|}^{(2)}(s)}{\cot(\rho\pi) + s \frac{d}{ds} \ln H_{|n-\mu|}^{(1)}(s)}, \quad (\text{A.4})$$

$$\Sigma_3(s, \varphi) = 2 \sum_{|n-\mu| > s} e^{in\varphi} e^{i(|n|-|n-\mu|)\pi} \frac{J_{|n-\mu|}(s)}{H_{|n-\mu|}^{(1)}(s)} \frac{\cot(\rho\pi) + s \frac{d}{ds} \ln J_{|n-\mu|}(s)}{\cot(\rho\pi) + s \frac{d}{ds} \ln H_{|n-\mu|}^{(1)}(s)}. \quad (\text{A.5})$$

The first part is easily calculated as a finite sum of geometric progression, yielding

$$\Sigma_1(s, \varphi) = 2\pi [\cos(\mu\pi) \Delta_s^{(\nu)}(\varphi) - \sin(\mu\pi) \Gamma_s^{(\nu)}(\varphi)], \quad (\text{A.6})$$

with $\Delta_s^{(\nu)}(\varphi)$ and $\Gamma_s^{(\nu)}(\varphi)$ given by (51)-(56).

The second part is presented as

$$\begin{aligned} \Sigma_2(s, \varphi) = & \sum_{n=\nu+1}^{s_+} e^{i(n\varphi+\mu\pi)} \frac{H_{n-\mu}^{(2)}(s)}{H_{n-\mu}^{(1)}(s)} \frac{\cot(\rho\pi) + s \frac{d}{ds} \ln H_{n-\mu}^{(2)}(s)}{\cot(\rho\pi) + s \frac{d}{ds} \ln H_{n-\mu}^{(1)}(s)} + \\ & + \sum_{n=-\nu}^{s_-} e^{-i(n\varphi+\mu\pi)} \frac{H_{n+\mu}^{(2)}(s)}{H_{n+\mu}^{(1)}(s)} \frac{\cot(\rho\pi) + s \frac{d}{ds} \ln H_{n+\mu}^{(2)}(s)}{\cot(\rho\pi) + s \frac{d}{ds} \ln H_{n+\mu}^{(1)}(s)}, \end{aligned} \quad (\text{A.7})$$

where $s_{\pm} = \llbracket s \pm \mu \rrbracket$. Using the asymptotics of the Hankel functions at $\alpha < s_{\min} \leq s < \infty$ (see, e.g., [13]):

$$H_{\alpha}^{(1)}(s) = \left(\frac{2e^{-i\pi/2}}{\pi} \right)^{1/2} (s^2 - \alpha^2)^{-1/4} \exp \left[i (s^2 - \alpha^2)^{1/2} - i\alpha \arccos(\alpha/s) \right] \times \\ \times [1 + O(s^{-1})], \quad (\text{A.8})$$

$$H_{\alpha}^{(2)}(s) = \left(\frac{2e^{i\pi/2}}{\pi} \right)^{1/2} (s^2 - \alpha^2)^{-1/4} \exp \left[-i (s^2 - \alpha^2)^{1/2} + i\alpha \arccos(\alpha/s) \right] \times \\ \times [1 + O(s^{-1})], \quad (\text{A.9})$$

we get

$$\Sigma_2(s, \varphi) = i \sum_{n=\nu+1}^{s_+} e^{2\pi i f_+(n,s)} + i \sum_{n=-\nu}^{s_-} e^{2\pi i f_-(n,s)} + O(s^{-1}), \quad (\text{A.10})$$

where

$$\begin{aligned} f_{\pm}(n, s) = & \pi^{-1} \left\{ -[s^2 - (n \mp \mu)^2]^{1/2} + (n \mp \mu) \arccos \left(\frac{n \mp \mu}{s} \right) - \right. \\ & \left. - \arctan \left(\frac{[s^2 - (n \mp \mu)^2]^{1/2}}{\cot(\rho\pi) - \frac{s^2}{2} [s^2 - (n \mp \mu)^2]^{-1}} \right) \pm \frac{1}{2} n\varphi \pm \frac{1}{2} \mu\pi \right\}. \end{aligned} \quad (\text{A.11})$$

Using an expansion into the Fourier series, one can get relation

$$\sum_{n=n_1}^{n_2} \exp[2\pi i f(n)] = \sum_{l \in \mathbb{Z}} \int_{n_1}^{n_2} du \exp\{2\pi i [f(u) - lu]\} + \frac{1}{2} \exp[2\pi i f(n_1)] + \frac{1}{2} \exp[2\pi i f(n_2)]. \quad (\text{A.12})$$

If $n_2 - n_1 \gg 1$ and $f(u)$ is convex upwards on interval $n_1 < u < n_2$, $\frac{d^2 f}{du^2} < 0$, then only a finite number of terms in the series on the right-hand side of (A.12) makes the main contribution, and one can use the method of stationary phase for its evaluation. Namely, let $s \gg 1$, $h(u)$ be a continuous function, and $g(u)$ be a real function that is convex upwards, $\frac{d^2 g}{du^2} < 0$, with one stationary point, $\frac{dg}{du}|_{u=u_0} = 0$, inside the interval, $n_1 < u_0 < n_2$. Then

$$\int_{n_1}^{n_2} du h(u) \exp[isg(u)] = h(u_0) \exp[isg(u_0)] \left[\frac{2\pi e^{i\pi/2}}{s(d^2 g/du^2)|_{u=u_0}} \right]^{1/2} + O(s^{-1}). \quad (\text{A.13})$$

In our case

$$h_{\pm}(u) = \exp \left\{ -2i \arctan \left(\frac{[s^2 - (u \mp \mu)^2]^{1/2}}{\cot(\pi) - \frac{s^2}{2} [s^2 - (u \mp \mu)^2]^{-1}} \right) \right\}, \quad (\text{A.14})$$

$$g_{\pm}^{(l)}(u) = \frac{2}{s} \left\{ -[s^2 - (u \mp \mu)^2]^{1/2} + (u \mp \mu) \arccos \left(\frac{u \mp \mu}{s} \right) \pm \frac{1}{2} u \varphi \pm \frac{1}{2} \mu \pi - ul\pi \right\}, \quad (\text{A.15})$$

and one gets

$$\frac{d}{du} g_{\pm}^{(l)} = \frac{2}{s} \left[\arccos \left(\frac{u \mp \mu}{s} \right) \pm \frac{1}{2} \varphi - l\pi \right], \quad (\text{A.16})$$

$$\frac{d^2}{du^2} g_{\pm}^{(l)} = -\frac{2}{s} [s^2 - (u \mp \mu)^2]^{-1/2}. \quad (\text{A.17})$$

Thus, $g_+^{(l)}(u)$ is convex upwards on interval $\nu + 1 < u < s_+$, and $g_-^{(l)}(u)$ is convex upwards on interval $-\nu < u < s_-$. Taking φ from the range $-\pi < \varphi < \pi$, one finds that, since $0 < u \mp \mu < s$, the stationary point inside the interval exists only at $l = 0$ and is determined by

$$\arccos \left(\frac{u_0 - \mu}{s} \right) = -\frac{1}{2} \varphi \quad \text{at} \quad -\pi < \varphi < 0 \quad (\text{A.18})$$

for $g_+^{(0)}(u)$ and by

$$\arccos \left(\frac{u_0 + \mu}{s} \right) = \frac{1}{2} \varphi \quad \text{at} \quad 0 < \varphi < \pi \quad (\text{A.19})$$

for $g_-^{(0)}(u)$; the integrals with $l \neq 0$ make a contribution of order $O(1)$. Consequently, we get

$$\begin{aligned} \sum_{n=\nu+1}^{s_+} e^{2\pi i f_+(n,s)} &= [-s\pi e^{i\pi/2} \sin(-\varphi/2)]^{1/2} \times \\ &\times \exp \left\{ -2i \arctan \left[\frac{2s \sin^3(-\varphi/2)}{2 \cot(\rho\pi) \sin^2(-\varphi/2) - 1} \right] \right\} \exp [-2is \sin(-\varphi/2) + i\mu(\varphi + \pi)] + \\ &+ O(1), \quad -\pi < \varphi < 0 \end{aligned} \quad (\text{A.20})$$

and

$$\begin{aligned} \sum_{n=-\nu}^{s_-} e^{2\pi i f_-(n,s)} &= [-s\pi e^{i\pi/2} \sin(\varphi/2)]^{1/2} \times \\ &\times \exp \left\{ -2i \arctan \left[\frac{2s \sin^3(\varphi/2)}{2 \cot(\rho\pi) \sin^2(\varphi/2) - 1} \right] \right\} \exp [-2is \sin(\varphi/2) + i\mu(\varphi - \pi)] + \\ &+ O(1), \quad 0 < \varphi < \pi. \end{aligned} \quad (\text{A.21})$$

Combining (A.20) and (A.21), we obtain

$$\begin{aligned} \Sigma_2(s, \varphi) &= [s\pi e^{i\pi/2} |\sin(\varphi/2)|]^{1/2} \exp [-2i\chi^{(\rho)}(s, \varphi)] \times \\ &\times \exp \{ -2is |\sin(\varphi/2)| + i\mu [\varphi - \text{sgn}(\varphi)\pi] \} + O(1), \quad -\pi < \varphi < \pi, \end{aligned} \quad (\text{A.22})$$

where $\chi^{(\rho)}(s, \varphi)$ is given by (50).

To evaluate the last part in (A.2), we rewrite it as

$$\begin{aligned} \Sigma_3(s, \varphi) &= 2 \sum_{n=s_++1}^{\infty} e^{i(n\varphi+\mu\pi)} \frac{J_{n-\mu}(s)}{H_{n-\mu}^{(1)}(s)} \frac{\cot(\rho\pi) + s \frac{d}{ds} \ln J_{n-\mu}(s)}{\cot(\rho\pi) + s \frac{d}{ds} \ln H_{n-\mu}^{(1)}(s)} + \\ &+ 2 \sum_{n=s_-+1}^{\infty} e^{-i(n\varphi+\mu\pi)} \frac{J_{n+\mu}(s)}{H_{n+\mu}^{(1)}(s)} \frac{\cot(\rho\pi) + s \frac{d}{ds} \ln J_{n+\mu}(s)}{\cot(\rho\pi) + s \frac{d}{ds} \ln H_{n+\mu}^{(1)}(s)}, \end{aligned} \quad (\text{A.23})$$

and use the asymptotics of the cylindrical functions at $s \leq s_{\max} < \alpha$ [13]:

$$J_\alpha(s) = (2\pi)^{-1/2} (\alpha^2 - s^2)^{-1/4} \exp \left[\sqrt{\alpha^2 - s^2} - \alpha \text{arccosh}(\alpha/s) \right] [1 + O(\alpha^{-1})], \quad (\text{A.24})$$

$$Y_\alpha(s) = - \left(\frac{2}{\pi} \right)^{1/2} (\alpha^2 - s^2)^{-1/4} \exp \left[-\sqrt{\alpha^2 - s^2} + \alpha \text{arccosh}(\alpha/s) \right] [1 + O(\alpha^{-1})], \quad (\text{A.25})$$

where $Y_\alpha(s)$ is the Neumann function of order α ($H_\alpha^{(1)}(s) = J_\alpha(s) + iY_\alpha(s)$). We get

$$\Sigma_3(s, \varphi) = i\Sigma_3^{(+)}(s, \varphi) + i\Sigma_3^{(-)}(s, \varphi), \quad (\text{A.26})$$

where

$$\begin{aligned} \Sigma_3^{(\pm)}(s, \varphi) = & \sum_{n=s_\pm+1}^{\infty} \exp \left[2\sqrt{(n \mp \mu)^2 - s^2} - 2(n \mp \mu) \operatorname{arccosh} \left(\frac{n \mp \mu}{s} \right) \right] \times \\ & \times \left[1 + F^{(\rho)} \left(\frac{n \mp \mu}{s}, s \right) \right]^{-1} e^{\pm i(n\varphi + \mu\pi)} \{ 1 + O[(n \mp \mu)^{-1}] \}, \end{aligned} \quad (\text{A.27})$$

$$F^{(\rho)}(v, s) = [e^{2i\kappa(v, s)} - 1] s \sqrt{v^2 - 1} \left[\cot(\rho\pi) + \frac{1}{2}(v^2 - 1)^{-1} + s \sqrt{v^2 - 1} \right]^{-1}, \quad (\text{A.28})$$

$$\kappa(v, s) = \arctan \left\{ 2 \exp \left[2s \left(v \operatorname{arccosh} v - \sqrt{v^2 - 1} \right) \right] \right\}. \quad (\text{A.29})$$

We evaluate sum (A.27) at $\varphi = 0$ as

$$\begin{aligned} \Sigma_3^{(\pm)}(s, 0) = & e^{\pm i\mu\pi} \int_{s_\pm+1}^{\infty} du \exp \left[2\sqrt{(u \mp \mu)^2 - s^2} - 2(u \mp \mu) \operatorname{arccosh} \left(\frac{u \mp \mu}{s} \right) \right] \times \\ & \times [1 + F^{(\rho)} \left(\frac{u \mp \mu}{s}, s \right)]^{-1} + O(1), \end{aligned} \quad (\text{A.30})$$

since the real part of the logarithmic derivative of the integrand in (A.30) is negative for $s_\pm + 1 < u < \infty$. Further, by substituting $u \mp \mu = s \cosh \tau$, we get

$$\begin{aligned} e^{\mp i\mu\pi} \Sigma_3^{(\pm)}(s, 0) = & s \int_0^{\infty} d\tau \sinh \tau \exp [2s(\sinh \tau - \tau \cosh \tau)] \times \\ & \times [1 + F^{(\rho)}(\cosh \tau, s)]^{-1} + O(1). \end{aligned} \quad (\text{A.31})$$

The integral in (A.31) is estimated with the use of the Laplace method, yielding

$$e^{\mp i\mu\pi} \Sigma_3^{(\pm)}(s, 0) = \Gamma \left(\frac{2}{3} \right) \left(\frac{s}{12} \right)^{1/3} + O(1) \quad (\text{A.32})$$

at $s \rightarrow \infty$, where $\Gamma(u)$ is the Euler gamma-function. Using the last relation, we get estimate

$$|\Sigma_3(s, \varphi)| \leq cs^{1/3}, \quad (\text{A.33})$$

where constant c is independent of s and φ .

Collecting (A.6), (A.22) and (A.33), and multiplying by $i(2\pi k)^{-1/2}$, we obtain (49).

Appendix B

Let us consider the right-hand side of (40):

$$\begin{aligned} \int_{-\pi}^{\pi} d\varphi |f_c(k, \varphi)|^2 &= \frac{4}{k} \sum_{n \in \mathbb{Z}} \left[\Upsilon_{|n-\mu|}^{(\rho)}(kr_c) \right]^* \Upsilon_{|n-\mu|}^{(\rho)}(kr_c) = \\ &= \frac{4}{k} \sum_{n \in \mathbb{Z}} \frac{\left\{ \left[\cot(\rho\pi) + u \frac{d}{du} \right] J_{|n-\mu|}(u) \right\}^2}{\left\{ \left[\cot(\rho\pi) + u \frac{d}{du} \right] J_{|n-\mu|}(u) \right\}^2 + \left\{ \left[\cot(\rho\pi) + u \frac{d}{du} \right] Y_{|n-\mu|}(u) \right\}^2} \Big|_{u=kr_c}. \end{aligned} \quad (\text{B.1})$$

In the short-wavelength limit, the sum is cut at $|n - \mu| = kr_c$ (see Appendix A), and the large-argument asymptotics for the cylindrical functions is used. As a result, we get the following expression for σ (43):

$$\begin{aligned} \sigma &= \frac{4}{k} \left\{ \left[\cot(\rho\pi) - \frac{1}{2} \right]^2 + k^2 r_c^2 \right\}^{-1} \times \\ &\times \sum_{|n-\mu| \leq kr_c} \left\{ \left[\cot(\rho\pi) - \frac{1}{2} \right] \cos \left(kr_c - \frac{1}{2} |n - \mu| \pi - \frac{\pi}{4} \right) - \right. \\ &\quad \left. - kr_c \sin \left(kr_c - \frac{1}{2} |n - \mu| \pi - \frac{\pi}{4} \right) \right\}^2. \end{aligned} \quad (\text{B.2})$$

Performing the summation, we get

$$\sigma = \frac{4\pi}{k} \left[\Delta_{kr_c}^{(\nu)}(0) + \cos(\mu\pi) \Delta_{kr_c}^{(\nu)}(\pi) A_1^{(\rho)}(kr_c) + i \sin(\mu\pi) \Gamma_{kr_c}^{(\nu)}(\pi) A_2^{(\rho)}(kr_c) \right], \quad (\text{B.3})$$

where

$$A_1^{(\rho)}(u) = \sin \left\{ 2u + 2 \arctan \left[\frac{2u}{2 \cot(\rho\pi) - 1} \right] \right\} \quad (\text{B.4})$$

and

$$A_2^{(\rho)}(u) = \cos \left\{ 2u + 2 \arctan \left[\frac{2u}{2 \cot(\rho\pi) - 1} \right] \right\}. \quad (\text{B.5})$$

Using (51)-(56), we get

$$\sigma = \frac{4}{k} s_c - \frac{2}{k} \sin(\mu\pi) e^{i\nu\pi} (1 - e^{is_c\pi}) A_2^{(\rho)}(kr_c) \quad (\text{B.6})$$

in the case when s_c is given by (53), or

$$\sigma = \frac{4}{k} (s_c + \frac{1}{2}) \pm \frac{2}{k} \cos(\mu\pi) e^{i\nu\pi} e^{is_c\pi} A_1^{(\rho)}(kr_c) - \frac{2}{k} \sin(\mu\pi) e^{i\nu\pi} A_2^{(\rho)}(kr_c) \quad (\text{B.7})$$

in the case when s_c is given by (56). Since s_c is estimated as $s_c = kr_c + O(1)$ in the short-wavelength limit, we get (58).

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